State Repression, Protest Diffusion, and Uncertainty about Regime Change

Jean Lachapelle *

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Abstract

For citizens living under authoritarian rule, protests aimed at regime change often raise hopes of a better future. But they are risky endeavors. Many popular movements that have toppled authoritarian regimes have failed to deliver on their promises, some with negative consequences for the groups that supported them. For citizens, this potential risk influences whether or not they choose to join protests, which in turn, affects the ruler's decision to use repression against them. This research note analyzes this dilemma and explains its significance for literature on the repression and dissent. I show that taking into account the uncertainty inherent in the consequences of regime change can simplify previous models of repression and revolt, generating a unique equilibrium and tractable predictions. This research note suggests that inconclusive empirical findings in existing scholarship may not be due to equilibrium selection, but rather the challenges of studying endogenous processes using observational data.

*Assistant Professor, Department of Political Science, Université de Montréal (email: jean.lachapelle.1@umontreal.ca). The author wishes to thank the participants of the Formal Models of Conflict Conference 2023 at the University of Pittsburgh, and the Western Political Science Association (WPSA) Mini-Conference on Undemocratic Politics 2024 for helpful comments on previous versions of this paper.

1 Introduction

State repression is known to backfire, causing strong negative reactions among ordinary citizens and increasing dissent rather than decreasing it. In contexts as diverse as the Soviet Union and the Philippines during the 1980s, as well as Egypt during the 2010s, violent repression has caused protests to spread, encouraging citizens to mobilize against the state.[1] The diffusion of protests to various sectors of society can be highly destabilizing for authoritarian regimes and lead to their overthrow (Karklins and Petersen, 1993; Lohmann, 1994; McAdam et al., 2001). Explaining when and how repression causes contention to spread is the subject of a rich scholarship. Overall, researchers find that repression's effects are difficult to predict and vary across contexts (Davenport, 2007, 8). This unpredictability may be related to the presence of multiple equilibria, which could account for repression's inconsistent effects (Shadmehr and Boleslavsky, 2022).

This research note draws attention to an important yet relatively overlooked aspect of such dynamics of contention, namely the fact that citizens are typically highly uncertain about the future that protests will bring about.[2] Protests may lead to revolutions that can enhance the well-being of citizens. However, such revolutions may also turn violently against the people and inflict harm, as shown in the cases of Iran in 1979 and Russia in 1917 (Shadmehr and Bernhardt, 2011, 832).[3] Moreover, revolutions can give rise to long-lasting authoritarian regimes that suppress individual rights (Lachapelle et al., 2020; Levitsky and Way, 2022; Meng and Paine, 2022). Disgruntled citizens who seek to join protests and contemplate revolution as a means of improving their well-being face a difficult choice, since it may bring to power a ruler who is even worse than the incumbent.

This research note analyzes the citizen's dilemma and its consequences the study of repression and revolt. I analyze the interaction of a ruler, who chooses whether or not to

[1]See Beissinger (2002, 354), Jumet (2018, 85-121), and Thompson (1995).
[2]See Shadmehr and Bernhardt (2011) for an important exception.
[3]See also Goldstone (2024).

repress a non-strategic challenger, and a bystanding citizen who chooses whether or not to join protests aimed at toppling the ruler after observing repression.[4] The bystander is uncertain about which outcome, regime overthrow or survival, will maximize his well-being; and repression conveys new information to the bystander about these potential futures. I build on the framework developed by Shadmehr and Boleslavsky (2022) who similarly analyze a model in which a bystander to repression chooses whether or not to revolt under conditions of uncertainty. I innovate by requiring the bystander to be *more uncertain about the consequences of protesting than those of acquiescing*. I demonstrate that this change makes the model more tractable and leads to more definite predictions. In the model in Shadmehr and Boleslavsky (2022), there are strategic complementarities between the ruler's decision to concede and the bystander's decision to revolt, leading to multiple equilibria. I show that, under plausible assumptions, these strategic complementarities disappear once we impose the restriction that the bystander is more uncertain about the consequences of protesting versus acquiescing; this leads to an equilibrium that is unique.[5] These results suggest that the inconclusive findings of the repression-protest literature may not be due to equilibrium selection. Rather they might arise from data limitations and the intrinsic empirical difficulties of studying such endogenous processes as repression and revolt.

2 Model

There is a ruler, a *bystander* (also called a *citizen*) and a non-strategic *challenger* (also called an *activist*). The non-strategic activist moves first and makes some policy demand. A ruler (pronoun she) chooses whether to repress or concede to the activist. If she concedes, she implements the activist's demand and the game ends. If she represses, the bystander

[4] In what follows, I use the terms bystander, citizen and bystanding citizen interchangeably.

[5] Interestingly, the strategic complementarities reappear and there are multiple equilibria once we relax this restriction.

(pronoun he) moves next and chooses whether to revolt or acquiesce. If the bystander revolts, the ruler is overthrown with some probability π, the activist seizes power and implements the activist's preferred policy.[6] If the citizen acquiesces, repression succeeds; the ruler remains in power and implements her ideal policy. The ruler has complete information, and the bystander has incomplete information about both the ruler and the activist.

This setup preserves the key features of Shadmehr and Boleslavsky (2022)'s model: the ruler has more information than the bystander, there is no protest after concession, and the activist's mobilization occurs exogenously.[7] The main technical difference is that I modify the belief structure to accommodate the assumption that the bystander is more uncertain about the consequences of regime change than those of regime survival. Specifically, I use continuous, instead of binary, player types and normally distributed beliefs. In Shadmehr and Boleslavsky (2022)'s model, actors are either good or bad, from the bystander's perspective. What makes the activist good is a preference for good reform that increases the bystander's welfare relative to the status quo. In contrast, a bad activist seeks the implementation of bad reform that reduces the bystander's welfare relative to the status quo. In turn, a good ruler prefers good reform to the status quo, whereas a bad ruler prefers the status quo to good reform.[8]

I consider player types as lying on a continuum. Let $\theta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ be the ruler's and activist's preferred policies, respectively; and let the bystander's preferred policy equal to

[6]For simplicity, Shadmehr and Boleslavsky (2022) assume that the ruler is overthrown with probability 1 whenever the bystander revolts. Their results do not depend on this modeling decision and remain the same if the ruler is overthrown with some positive probability smaller than 1. See Shadmehr and Boleslavsky (2022, 155). I assume that revolt succeeds with some positive probability smaller than 1 to avoid the unrealistic scenario in which a ruler chooses to repress knowing that doing so will inevitably cause her to be overthrown. This is not a substantive modeling decision and the results are the same if revolt succeeds with probability 1.

[7]See Shadmehr and Boleslavsky (2022) for detailed justifications of these assumptions.

[8]To make the problem tractable, the authors assume that the ruler never prefers bad reform to good reform. They also assume that the bad ruler's conflict of interest with the good activist is greater than the good ruler's conflict of interest with the bad activist Shadmehr and Boleslavsky (2022, 152).

zero without loss of generality. The bystander's priors $\hat{\theta}$ and $\hat{\gamma}$ are described by independent normal distributions,

$$\begin{pmatrix} \hat{\theta} \\ \hat{\gamma} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_\theta \\ \mu_\gamma \end{pmatrix}, \begin{pmatrix} \sigma_\theta & 0 \\ 0 & \sigma_\gamma \end{pmatrix}\right) \tag{1}$$

where μ_θ and μ_γ are the preferred policies of the ruler and activist, respectively; σ_θ and σ_γ are the bystander's uncertainty about the ruler and the activist, respectively.[9] Note that the condition $\sigma_\gamma > \sigma_\theta$ corresponds to a scenario in which the bystander is more uncertain about the consequences of regime overthrow than those of regime survival, whereas $\sigma_\gamma < \sigma_\theta$ describes the opposite scenario in which a bystander is more uncertain about the consequences of regime survival than of regime overthrow. The former scenario is the one that is most realistic, thus we set $\sigma_\gamma > \sigma_\theta$.[10]

Table 1: Payoffs

Outcome	Ruler's Payoff	Citizen's Payoff
Repression Fails	$-c-t$	$-\gamma^2$
Repression Succeeds	$-c$	$-\theta^2$
Ruler Concedes	$-(\theta-\gamma)^2$	$-\gamma^2$

Payoffs are quadratic in the policy distance. The ruler's payoff in the outcome in which she concedes to the challenger is $-(\theta-\gamma)^2$. Her payoff in the outcome in which she represses and repression succeeds (either because a bystander acquiesced or unsuccessfully revolted) is $-c$, where c is the direct cost of repression.[11] Finally, the ruler's payoff in the outcomes in which she represses and repression fails (because a bystander successfully revolted) is $-c-t$,

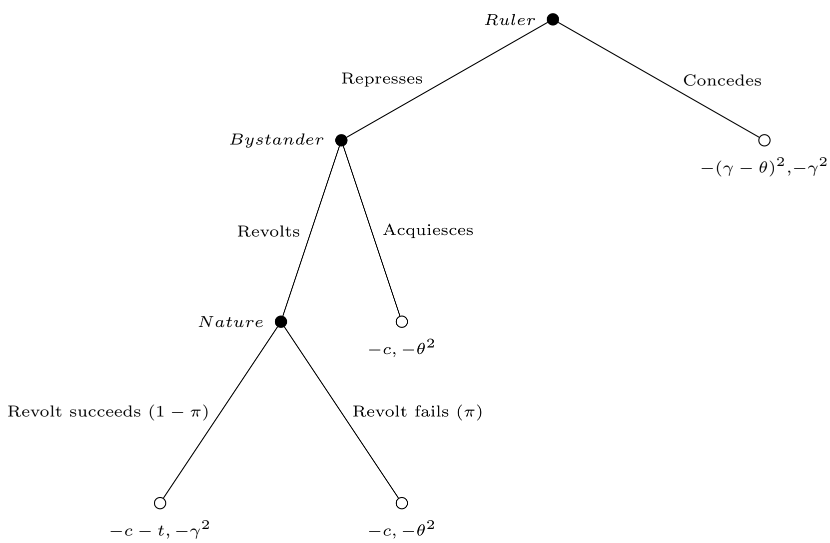
[9]The values of μ_θ, μ_γ, σ_θ and σ_γ are common knowledge.

[10]In a model where player types are binary, it is cumbersome to impose the restriction that the bystander is more uncertain about the consequences of regime overthrow than those of regime survival. Consider a situation in which a ruler is bad with probability $p \in (0,1)$ and an activist is bad with probability $q \in (0,1)$. The variances in outcomes are $p(1-p)$ and $q(1-q)$, and imposing the restriction involves respecting the inequality $p(1-p) < q(1-q)$, which is non-linear in the model parameters.

[11]

where t denotes the "turnover" cost of losing power.[12] The bystander's payoff in the outcome in which the ruler represses the activist and repression succeeds is $-\theta^2$. If she represses the activist and repression fails, the bystander's payoff is $-\gamma^2$. Finally, the bystander's payoff when the ruler concedes to the activist is $-\gamma^2$. The probability that revolt succeeds, $1-\pi$, is common knowledge. Table 1 summarizes the players' payoffs in the different outcomes, while Figure 1 presents the steps of the game.

Figure 1: Game Tree



2.1 Best Responses

A ruler's strategy is a mapping from her ideal policy θ and the activist's demanded policy γ to a probability of repression, $s_r : \mathbb{R}^2 \to [0, 1]$. A bystander's strategy is a probability $s_b \in [0, 1]$ of revolt. The solution concept is Perfect Bayesian Equilibrium.

A ruler's best response is a cutoff strategy

[12] The parameters c and t are common knowledge.

$$s_r = \begin{cases} 1 \text{ if } |\theta - \gamma| > \kappa(s_b), \\ 0 \text{ if } |\theta - \gamma| < \kappa(s_b), \\ [0,1] \text{ if } |\theta - \gamma| = \kappa(s_b), \end{cases} \tag{2}$$

where $\kappa(s_b) = \sqrt{s_b(1-\pi)t + c}$ is the cost of repression. Note that κ is increasing in s_b: as the bystander revolts more often, the ruler has increasing incentives to concede.

Repression reveals the existence of a conflict between the ruler and the activist, $|\theta - \gamma| > \kappa(s_b)$. Denote a bystander's marginal payoff of revolting, by $\Delta = \mathbb{E}[\hat{\theta}^2 - \hat{\gamma}^2 | \kappa < |\theta - \gamma|]$. A bystander's best response is

$$s_b = \begin{cases} 1 \text{ if } \Delta(\kappa) > 0, \\ 0 \text{ if } \Delta(\kappa) < 0, \\ [0,1] \text{ if } \Delta(\kappa) = 0, \end{cases} \tag{3}$$

Proposition 1 (In equilibrium, the bystander's revolt is weakly decreasing in κ.)**.** *If the bystander ex-ante prefers the ruler, $\mu_\theta^2 + \sigma_\theta^2 < \mu_\gamma^2 + \sigma_\gamma^2$, the bystander's best response is $s_b = 0$ for any value of κ. If the bystander ex-ante prefers the challenger, $\mu_\theta^2 + \sigma_\theta^2 > \mu_\gamma^2 + \sigma_\gamma^2$, then there exists $\kappa^\dagger > 0$ such that the bystander's best response is $s_b = 1$ whenever $\kappa < \kappa^\dagger$, $s_b = 0$ whenever $\kappa > \kappa^\dagger$; and the bystander is indifferent between revolting and acquiescing whenever $\kappa = \kappa^\dagger$.*

Proof in the appendix.

Proposition 1 is the main result behind the unicity of the equilibrium. It implies that more frequent repression (weakly) increases the bystander's incentives to revolt. Yet, since *more revolt* decreases the ruler's incentives to repress, there are no strategic complementarities between the ruler and the bystander's decision; thus the multiple equilibria that characterized previous models disappear (see Lemma 3 in the Appendix).

2.2 Equilibrium Characterization

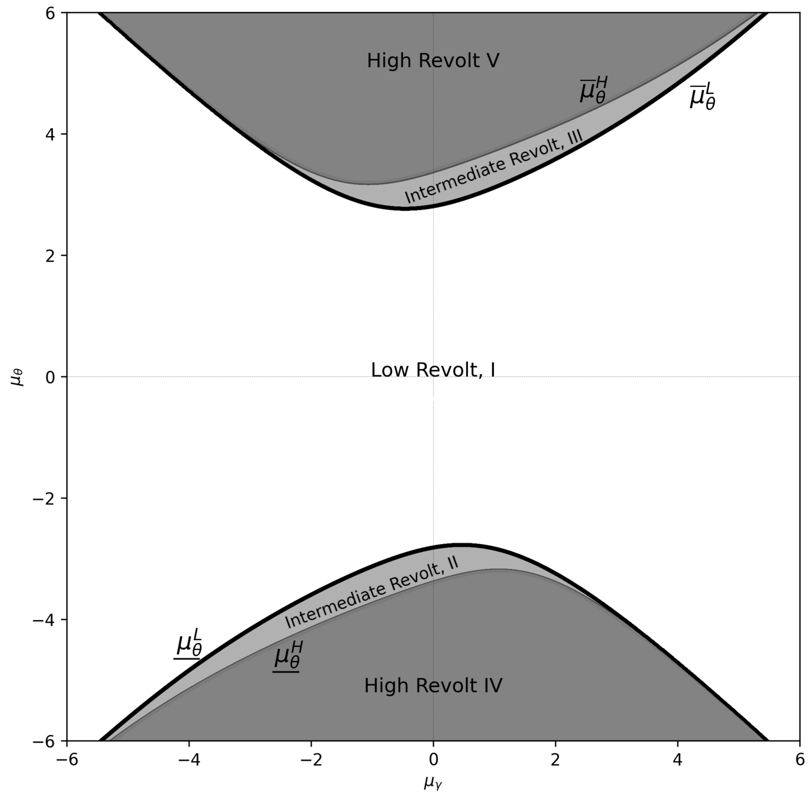
There are four curves, $\underline{\mu}_\theta^H(\mu_\gamma) < \underline{\mu}_\theta^L(\mu_\gamma) < 0 < \overline{\mu}_\theta^L(\mu_\gamma) < \overline{\mu}_\theta^H(\mu_\gamma)$, that demarcate regions of low, intermediate and high revolt. In Region I ($\underline{\mu}_\theta^L < \mu_\theta < \overline{\mu}_\theta^L$), revolt is *low*; the bystander never revolts, $s_b^* = 0$, and the ruler employs cutoff rule $\kappa^* = \kappa^L = \sqrt{c}$. In Regions II ($\underline{\mu}_\theta^H < \mu_\theta < \underline{\mu}_\theta^L$) and III ($\overline{\mu}_\theta^L < \mu_\theta < \overline{\mu}_\theta^H$), revolt is *intermediate*; the bystander revolts with probability $s_b^* = \frac{(\kappa^\dagger)^2 - c}{(1-\pi)t} \in (0,1)$, where the ruler employs cutoff rule $\kappa^* = \kappa^\dagger \in (\sqrt{c}, \sqrt{c + (1-\pi)t})$. In Regions IV ($\mu_\theta < \underline{\mu}_\theta^H$) and V ($\mu_\theta > \overline{\mu}_\theta^H$), revolt is *high*; the bystander revolts with probability $s_b^* = 1$, and the ruler employs cutoff rule $\kappa^* = \kappa^H = \sqrt{c + (1-\pi)t}$.

Figure 2 illustrates the equilibrium structure, showing the five regions in the μ_γ, μ_θ-plane.[13] A horizontal (vertical) move away from the origin indicates that a bystander's (*ex-ante*) expected conflict with the activist (ruler) increases. As expected, the bystander revolts more often when he dislikes the ruler more, keeping μ_γ constant. In Region I, $|\mu_\theta|$ is small and the bystander does not revolt in equilibrium (Low Revolt). As μ_θ increases in absolute value (keeping μ_γ constant), we enter Region II or III (Intermediate Revolt), depending on the sign of μ_θ. In this region the bystander revolts with probability $s_b^* \in (0,1)$. Once we cross into Region IV or V (High Revolt), which is below (above) the curve $\underline{\mu}_\theta^H$ ($\overline{\mu}_\theta^H$), the bystander revolts with probability 1.

Why is the equilibrium unique in this model? The reason is that, as the ruler concedes more often, the bystander has increasing incentives to choose the action that minimizes his (ex-ante) uncertainty. Indeed, when κ is large (such that repression is rare and unexpected from the bystander's perspective), the sign of Δ equals the sign of $\sigma_\theta - \sigma_\gamma$. If instead we impose the less realistic condition $\sigma_\theta > \sigma_\gamma$, then the bystander's incentive to revolt weakly increases, as the ruler concedes more often, leading to strategic complementarities and multiple equilibria. By discarding the $\sigma_\theta > \sigma_\gamma$ region, we remove these strategic complementarities.

[13] For this illustration, the parameters are set to $\sigma_\theta = 1$, $\sigma_\gamma = 2$, $c = 12$, $t = 22$, $\pi = 0.35$.

Figure 2: Equilibrium Characterization



3 Conclusion

A revolution raises the hopes of a better future. But it is a risky endeavor. Many revolutions have failed to deliver on their promise of emancipation, and some have had negative consequences for the groups that supported them. This potential risk influences citizens' decision to join protests and overthrow their government, affecting in turn the ruler's decision to use repression against them. I have shown that accounting for the uncertainty inherent in the consequences of regime change can simplify models of repression and revolt. Building on the work of Shadmehr and Boleslavsky (2022), I have demonstrated that the multiple equilibria disappear once we impose the condition that citizens are more uncertain about the consequences of regime overthrow than those of regime survival. The results suggest a plausible interpretation of the "punishment puzzle," the observation that repression has inconsistent effects on mobilization. These inconsistent effects may be due to data and methodological limitations, rather than equilibrium selection. Future work may explore whether similar restrictions simplify the equilibrium structure of a broader range of models.

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Supplementary Material

Posterior Expectations

Let the ruler play a cutoff strategy that consists of repressing whenever $|\theta - \gamma| > \kappa$. After observing repression, a citizen updates using Bayes' rule

$$f_{\hat{\theta}}(x; \kappa, \text{repression}) = \frac{\left[1 - \Phi\left(\frac{x+\kappa-\mu_\gamma}{\sigma_\gamma}\right) + \Phi\left(\frac{x-\kappa-\mu_\gamma}{\sigma_\gamma}\right)\right] \frac{1}{\sigma_\theta}\phi\left(\frac{x-\mu_\theta}{\sigma_\theta}\right)}{R}$$

$\phi(\cdot)$ is the standard normal pdf, $\Phi(\cdot)$ is the standard normal cdf, and

$$R \equiv \int_{-\infty}^{\infty} \left[1 - \Phi\left(\frac{x+\kappa-\mu_\gamma}{\sigma_\gamma}\right) + \Phi\left(\frac{x-\kappa-\mu_\gamma}{\sigma_\gamma}\right)\right] \frac{1}{\sigma_\theta}\phi\left(\frac{x-\mu_\theta}{\sigma_\theta}\right) dx$$

is the probability of repression.

Let $z = \frac{x-\mu_\theta}{\sigma_\theta}$. The expression becomes

$$R = 1 - \int_{-\infty}^{\infty} \phi(z)\Phi\left(\frac{z\sigma_\theta - \kappa + \mu_\theta - \mu_\gamma}{\sigma_\gamma}\right) dz + \int_{-\infty}^{\infty} \phi(z)\Phi\left(\frac{z\sigma_\theta + \kappa + \mu_\theta - \mu_\gamma}{\sigma_\gamma}\right) dz$$

Using the identity $\int_{-\infty}^{\infty} \phi(x)\Phi(a+bx)dx = \Phi\left(\frac{a}{\sqrt{1+b^2}}\right)$ from Owen (1980, 403), we obtain

$$R = 1 - \Phi\left(\frac{\mu_\theta - \mu_\gamma + \kappa}{\sigma}\right) + \Phi\left(\frac{\mu_\theta - \mu_\gamma - \kappa}{\sigma}\right),$$

where $\sigma \equiv \sqrt{\sigma_\theta^2 + \sigma_\gamma^2}$.

To evaluate $\mathbb{E}[\hat{\theta}^2 | |\theta - \gamma| > \kappa]$, we need to compute the integrals $I_0 = \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma_\theta}\phi\left(\frac{x-\mu_\theta}{\sigma_\theta}\right) dx$ and $I_\pm = \int_{-\infty}^{\infty} x^2 \Phi(\frac{x-\mu_\gamma \pm \kappa}{\sigma_\gamma})\frac{1}{\sigma_\theta}\phi\left(\frac{x-\mu_\theta}{\sigma_\theta}\right) dx$. The first integral, I_0, is simply $\mu_\theta^2 + \sigma_\theta^2$. To compute $I_\pm$, let $z = (x-\mu_\theta)/\sigma_\theta$ such that $I_\pm = \int_{-\infty}^{\infty} (z^2\sigma_\theta^2 + 2z\mu_\theta\sigma_\theta + \mu_\theta^2)\Phi(\frac{z\sigma_\theta + \mu_\theta - \mu_\gamma \pm \kappa}{\sigma_\gamma})\phi(z)\, dz$. We will use the following identities derived from Owen (1980, 403-404):

$$\int_{-\infty}^{\infty} x\phi(x)\Phi(a+bx)dx = \frac{b}{\sqrt{1+b^2}}\phi\left(\frac{a}{\sqrt{1+b^2}}\right)$$
$$\int_{-\infty}^{\infty} x^2\phi(x)\Phi(a+bx)dx = \Phi\left(\frac{a}{\sqrt{1+b^2}}\right) - \frac{ab^2}{(1+b^2)^{3/2}}\phi\left(\frac{a}{\sqrt{1+b^2}}\right).$$

This gives

$$I_\pm = (\mu_\theta^2+\sigma_\theta^2)\Phi\left(\frac{\mu_\theta-\mu_\gamma\pm\kappa}{\sigma}\right) + \frac{\sigma_\theta^4}{\sigma^3}\left[\mu_\theta\left(1+2\left(\frac{\sigma_\gamma}{\sigma_\theta}\right)^2\right)+\mu_\gamma\mp\kappa\right]\phi\left(\frac{\mu_\theta-\mu_\gamma\pm\kappa}{\sigma}\right)$$

.

The sum $I_+ + I_-$ involves expressions of the form $\phi(\frac{\mu_\theta-\mu_\gamma+\kappa}{\sigma}) \pm \phi(\frac{\mu_\theta-\mu_\gamma-\kappa}{\sigma})$. This simplifies as:

$$\phi\left(\frac{\mu_\theta-\mu_\gamma+\kappa}{\sigma}\right)+\phi\left(\frac{\mu_\theta-\mu_\gamma-\kappa}{\sigma}\right) = -\sigma R'$$
$$\phi\left(\frac{\mu_\theta-\mu_\gamma+\kappa}{\sigma}\right)-\phi\left(\frac{\mu_\theta-\mu_\gamma-\kappa}{\sigma}\right) = \sigma R' \tanh\left(\frac{\kappa(\mu_\theta-\mu_\gamma)}{\sigma^2}\right) \tag{4}$$

where $R' \equiv \partial R/\partial\kappa < 0$, and $\tanh(\cdot)$ is the hyperbolic tangent function. Combining the terms gives the posterior expectations given $|\theta-\gamma|>\kappa$,

$$\mathbb{E}[\hat{\theta}^2|\kappa<|\theta-\gamma|] = \mu_\theta^2+\sigma_\theta^2 - \frac{\sigma_\theta^4 A}{\sigma^2}\cdot\frac{\partial \ln R}{\partial\kappa} \tag{5}$$
$$\mathbb{E}[\hat{\gamma}^2|\kappa<|\theta-\gamma|] = \mu_\gamma^2+\sigma_\gamma^2 - \frac{\sigma_\gamma^4 B}{\sigma^2}\cdot\frac{\partial \ln R}{\partial\kappa}. \tag{6}$$

where

$$A = \left(\mu_\theta + 2\mu_\theta \left(\frac{\sigma_\gamma}{\sigma_\theta}\right)^2 + \mu_\gamma\right) \tanh\left(\frac{\kappa(\mu_\theta - \mu_\gamma)}{\sigma^2}\right) + \kappa$$

$$B = -\left(\mu_\theta + 2\mu_\gamma \left(\frac{\sigma_\theta}{\sigma_\gamma}\right)^2 + \mu_\gamma\right) \tanh\left(\frac{\kappa(\mu_\theta - \mu_\gamma)}{\sigma^2}\right) + \kappa$$

Denote the citizen's marginal utility of revolt after observing repression, by $\Delta = \mathbb{E}[\hat{\theta}^2 - \hat{\gamma}^2 | \kappa < |\theta - \gamma|]$. From Equation (5), we get

$$\Delta = \mu_\theta^2 - \mu_\gamma^2 + \sigma_\theta^2 - \sigma_\gamma^2 - C\frac{\partial \ln R}{\partial \kappa}, \tag{7}$$

where

$$C = \sigma^2(\mu_\theta + \mu_\gamma) \tanh\left(\frac{\kappa(\mu_\theta - \mu_\gamma)}{\sigma^2}\right) + \underbrace{(\sigma_\theta^2 - \sigma_\gamma^2)}_{<0}\kappa, \tag{8}$$

and $\sigma_\theta^2 - \sigma_\gamma^2 < 0$ encodes the restriction that the bystander is uncertain about the outcome of revolution.

Note that when $\kappa = 0$, the ruler represses with probability 1: repression is uninformative, and the expectations of the posteriors equal the priors. We will use $\Delta(\kappa = 0) = \mu_\theta^2 - \mu_\gamma^2 + \sigma_\theta^2 - \sigma_\gamma^2$ to refer to the bystander's ex-ante marginal benefit of revolting.

Definition 1. A differentiable function $f(x)$ with domain $[0, \infty)$ is up-down if there exists $d \geq 0$ such that $f' > 0$ whenever $x < d$ and $f' < 0$ whenever $x > d$. Likewise, $f(x)$ is down-up if there exists $d > 0$ such that $f' < 0$ whenever $x < d$ and $f' > 0$ whenever $x > d$.[14]

Lemma 1. *$C(\kappa)$ is up-down. Moreover, if $\Delta(0) < 0$, then $C(\kappa)$ is strictly negative and decreasing. If $\Delta(0) > 0$, then $C(\kappa)$ has a unique root in $(0, \infty)$.*

[14] Note that a function $f(x)$ that is increasing (decreasing) over its entire domain is down-up (up-down).

Proof. Consider the function

$$\tilde{C}(\kappa) = \sigma^2 \frac{(\mu_\theta + \mu_\gamma)}{\sigma_\theta^2 - \sigma_\gamma^2} \tanh\left(\frac{\kappa(\mu_\theta - \mu_\gamma)}{\sigma^2}\right) + \kappa. \tag{9}$$

This function has the form $a \tanh(bx) + x$, which is down-up. Since $\sigma_\theta^2 - \sigma_\gamma^2 < 0$, this implies that $C(\kappa)$ is up-down. Moreover, $\tilde{C}(\kappa^\dagger) = 0 \iff C(\kappa^\dagger) = 0$, so they share the same roots. Note that the function $a \tanh(bx) + x$ has a root in $(0, \infty)$ if and only if $ab < -1$; otherwise, it is strictly increasing and positive. The condition $ab < -1$ holds if and only if $\frac{\mu_\theta^2 - \mu_\gamma^2}{\sigma_\theta^2 - \sigma_\gamma^2} < -1$ Since $\sigma_\theta^2 - \sigma_\gamma^2 < 0$, this inequality is equivalent to $\mu_\theta^2 - \mu_\gamma^2 > \sigma_\gamma^2 - \sigma_\theta^2$, which in turn is equivalent to $\mu_\theta^2 + \sigma_\theta^2 - \mu_\gamma^2 - \sigma_\gamma^2 > 0$. This means that $C(\kappa)$ has a unique root in $(0, \infty)$ if and only if $\mu_\theta^2 + \sigma_\theta^2 - \mu_\gamma^2 - \sigma_\gamma^2 > 0$. This corresponds to $\Delta(0) > 0$. □

Lemma 2. *R is log-concave in κ, that is, $\partial^2 \ln R/\partial\kappa^2 < 0$ for all $\kappa \geq 0$.*

Proof. Proving that R is log-concave is equivalent to proving that $r = \frac{R'}{R}$ is decreasing over the interval $\kappa \in [0, \infty)$. We will use L'Hôpital's rule for monotonicity (Pinelis 2006). Let $f = R'$ and $g = R$ and consider the function $\alpha = \frac{f'}{g'} = \frac{R''}{R'}$. The first step is to show that α is either up-down or strictly decreasing. We prove this by showing that α is concave. Note that

$$\frac{\partial^2 R}{\partial\kappa^2} = -\frac{1}{\sigma^2}\frac{\partial R}{\partial\kappa}\left(\kappa - (\mu_\theta - \mu_\gamma)\tanh\left(\frac{(\mu_\theta - \mu_\gamma)\kappa}{\sigma^2}\right)\right). \tag{10}$$

Thus

$$\alpha = \frac{1}{\sigma^2}\left[-\kappa + (\mu_\theta - \mu_\gamma)\tanh\left(\frac{(\mu_\theta - \mu_\gamma)\kappa}{\sigma^2}\right)\right], \tag{11}$$

so that

$$\frac{\partial \alpha}{\partial \kappa} = -\frac{1}{\sigma^2} + \frac{1}{\sigma^4}(\mu_\theta - \mu_\gamma)^2 \left(1 - \tanh^2\left(\frac{(\mu_\theta - \mu_\gamma)\kappa}{\sigma^2}\right)\right), \tag{12}$$

and

$$\frac{\partial^2 \alpha}{\partial \kappa^2} = \underbrace{\frac{-2(\mu_\theta - \mu_\gamma)^3}{\sigma^6} \tanh\left(\frac{(\mu_\theta - \mu_\gamma)\kappa}{\sigma^2}\right)}_{<0} \underbrace{\left[1 - \tanh^2\left(\frac{(\mu_\theta - \mu_\gamma)\kappa}{\sigma^2}\right)\right]}_{>0}, \tag{13}$$

which shows that α is concave.

Note that $\lim_{\kappa\to\infty} \alpha' = -1/\sigma^2 < 0$, so that α is decreasing for κ large. Since α is concave, this implies that α is either decreasing or up-down. From Proposition 4.1 of Pinelis (2006), if α is decreasing then r is decreasing, implying that R is log-concave.

Assume that α is up-down. Since both f and g go to zero as $\kappa \to \infty$, Proposition 4.3 in Pinelis (2006) applies, and therefore r is either increasing or up-down. Yet, r cannot be increasing since the derivative of r evaluated at zero is $r'(0) = \frac{R''(0)}{R(0)} - \frac{(R'(0))^2}{R(0)^2} = -\frac{(R'(0))^2}{R(0)^2} < 0$ (given that $R''(0) = 0$ from Equation (10)). Hence r must be up-down. The only way that r can be up-down is if it is decreasing over the entire interval $\kappa > 0$.[15] Therefore, $r = \frac{R'}{R}$ is decreasing, which implies that $\ln R$ is concave.

□

Proof of Proposition 1

Proof. We will show that if $\Delta(0) < 0$, then $\Delta(\kappa) < 0$ for all $\kappa \geq 0$. If $\Delta(0) > 0$, then there exists a unique $\kappa^\dagger$ such that $\Delta > 0$ for all $\kappa < \kappa^\dagger$, and $\Delta < 0$ for all $\kappa > \kappa^\dagger$.

If $\Delta(0) < 0$, then C is negative by Lemma 1 and thus Δ is also negative.

If $\Delta(0) > 0$, we show that Δ has at least one root by the intermediate value theorem.

[15]Since a strictly decreasing function is a special case of an up-down function from Definition A.1.

Take the limit

$$\lim_{\kappa\to\infty} \Delta = \mu_\theta^2 - \mu_\gamma^2 + \sigma_\theta^2 - \sigma_\gamma^2 + \lim_{\kappa\to\infty}\left(-C\frac{R'}{R}\right), \tag{14}$$

Recall that $R'' = -\frac{1}{\sigma^2}R'\left(\kappa - (\mu_\theta^2 - \mu_\gamma^2)\tanh\left(\frac{(\mu_\theta-\mu_\gamma)\kappa}{\sigma^2}\right)\right)$ from Equation (10), and apply l'Hôpital's rule:

$$\lim_{\kappa\to\infty} \Delta = \mu_\theta^2 - \mu_\gamma^2 + \sigma_\theta^2 - \sigma_\gamma^2 + \lim_{\kappa\to\infty}\left[-C' + \frac{1}{\sigma^2}C\left(\kappa - (\mu_\theta^2 - \mu_\gamma^2)\tanh\left(\frac{(\mu_\theta - \mu_\gamma)\kappa}{\sigma^2}\right)\right)\right]. \tag{15}$$

Since both C' and the hyperbolic tangent $\tanh(\cdot)$ are bounded functions, the term that dominates the limit is $\frac{1}{\sigma^2}\kappa C$. This term goes to minus infinity as $\kappa \to \infty$; therefore $\lim_{\kappa\to\infty}\Delta = -\infty$. Since the limit is $-\infty$, a root must exist by the intermediate value theorem.

Uniqueness is proven by contradiction. From Lemma 1, there exists $a > 0$ such that $C > 0$ on $\kappa \in [0, a)$, and $C < 0$ on $\kappa \in (a, \infty)$. Assume that Δ has more than one root. Then, Δ must cross zero at least twice. By Rolle's Theorem, there exists a local minimum $e > a$ such that $\Delta(e) < 0$. The derivative of Δ evaluated at e is

$$\Delta'(e) = -\underbrace{\frac{\partial^2 \ln R}{\partial\kappa^2}}_{<0}\underbrace{C(e)}_{<0} - \underbrace{\frac{\partial \ln R}{\partial\kappa}}_{<0}\underbrace{C'(e)}_{<0}. \tag{16}$$

The first term is negative because of the log-concavity of R and the fact that $C(\kappa) < 0$ for $\kappa > a$. The second term is negative because $R' < 0$ for all κ, and because C being up-down implies that $C'(\kappa) < 0$ for all $\kappa > a$. Thus $\Delta'(e) < 0$, which contradicts the statement that e is a local minimum. This completes the proof. □

Lemma 3. *The game has a unique Perfect Bayesian Equilibrium.*

Proof. If $\Delta(0) < 0$, then Δ is negative for all κ. The bystander's best response is $s_b^* = 0$; the ruler's best response is the cutoff rule $\kappa^* = \sqrt{c}$.

If $\Delta(0) > 0$, there exists $\kappa^\dagger$ such that $\Delta > 0$ for $\kappa < \kappa^\dagger$, $\Delta < 0$ for $\kappa > \kappa^\dagger$, and $\Delta(\kappa^\dagger) = 0$. There are three scenarios to consider.

1. If $\sqrt{c} < \sqrt{c+t} < \kappa^\dagger$, $\Delta > 0$ for any $\kappa \in [\sqrt{c}, \sqrt{c+t}]$. The bystander's best response is $s_b^* = 1$, and the ruler's best response is the cutoff rule $\kappa^* = \sqrt{c+t}$;

2. If $\kappa^\dagger < \sqrt{c} < \sqrt{c+t}$, $\Delta < 0$ for any $\kappa \in [\sqrt{c}, \sqrt{c+t}]$. The bystander's best response is $s_b^* = 0$; and the ruler's best response is the cutoff rule $\kappa^* = \sqrt{c}$;

3. If $\sqrt{c} < \kappa^\dagger < \sqrt{c+t}$, then the ruler can make the bystander indifferent between revolting and acquiescing by setting $\kappa^* = \kappa^\dagger$. In turn, $\kappa^* = \kappa^\dagger$ is a best response to $s_b^* = \frac{(\kappa^\dagger)^2 - c}{(1-\pi)t}$. Hence there is an equilibrium in which the bystander revolts with probability $s_b^* = \frac{(\kappa^\dagger)^2 - c}{(1-\pi)t}$ and the ruler uses the cutoff rule $\kappa^* = \kappa^\dagger$. Uniqueness follows by contradiction. Imagine that the bystander revolts with some probability $p_1 < \frac{(\kappa^\dagger)^2 - c}{(1-\pi)t}$. Then the ruler's best response to bystander strategy p_1 is to use a cutoff rule $\kappa_1 < \kappa^\dagger$. However, $\kappa_1 < \kappa^\dagger \implies \Delta(\kappa_1) > 0 \implies$ the bystander's best response is to revolt with probability 1. Thus p_1 is not a bystander strategy in equilibrium. Likewise, assume that the bystander revolts with some probability $p_2 > \frac{(\kappa^\dagger)^2 - c}{(1-\pi)t}$. Then the ruler's best response to p_2 is to use a cutoff rule $\kappa_2 > \kappa^\dagger$. However, $\kappa_2 > \kappa^\dagger \implies \Delta(\kappa_2) < 0 \implies$ the bystander's best response is to acquiesce with probability 1. Any deviation from the strategy $s_b^* = \frac{(\kappa^\dagger)^2 - c}{(1-\pi)t}$ is not a best response to the ruler's best response. The strategy pair $s_b^* = \frac{(\kappa^\dagger)^2 - c}{(1-\pi)t}$, $\kappa^* = \kappa^\dagger$ is the unique equilibrium.

$\square$

Equilibrium Characterization

From Lemma 3 (equilibrium uniqueness), the parameter space divides into regions of low, intermediate and high repression, which correspond to high, intermediate and low revolt respectively. In this section, I show that these regions are characterized by four curves in the μ_γ, μ_θ-plane.

Define rotated coordinates $\mu_+ = \mu_\theta + \mu_\gamma$ and $\mu_- = \mu_\theta - \mu_\gamma$, so that

$$\Delta = \mu_+\mu_- + \sigma_\theta^2 - \sigma_\gamma^2 - C\frac{\partial \ln R}{\partial \kappa}, \tag{17}$$

and $C = \sigma^2\mu_+ \tanh\left(\frac{\kappa\mu_-}{\sigma^2}\right) + (\sigma_\theta^2 - \sigma_\gamma^2)\kappa$. Note that R depends on μ_- but does not depend on μ_+. Hence Δ is monotonically increasing in μ_+ whenever $\mu_- > 0$ and monotonically decreasing whenever $\mu_- < 0$ (when $\mu_- = 0$, Δ does not depend on μ_+).

Let $\mu \neq 0$. Then

$$\mu_+^* = \frac{\left(\sigma_\gamma^2 - \sigma_\theta^2\right)\left(1 - \kappa\frac{\partial \ln R}{\partial \kappa}\right)}{\mu_- - \sigma^2 \tanh\left(\frac{\kappa\mu_-}{\sigma^2}\right)\frac{\partial \ln R}{\partial \kappa}} \tag{18}$$

$\Delta = 0$, whenever $\mu_+ = \mu_+^*$, holding other parameters constant. Note that μ_+^* is positive for μ_- positive and negative for $\mu_- < 0$ negative, with a discontinuity at $\mu_- = 0$.[16] Let $\mu_- > 0$. Then Δ is positive whenever $\mu_+ > \mu_+^*$ and negative whenever $\mu_+ < \mu_+^*$. Conversely, when $\mu_- < 0$, then Δ is negative whenever $\mu_+ > \mu_+^*$ and positive whenever $\mu_+ < \mu_+^*$.

We will show that μ_+^* is increasing in κ when $\mu_- < 0$, and decreasing in κ when $\mu_- > 0$. To demonstrate this, we use the implicit function theorem: $\frac{\partial \mu_+^*}{\partial \kappa} = -\left.\frac{\partial\Delta/\partial\kappa}{\partial\Delta/\partial\mu_+}\right|_{\mu_+ = \mu_+^*}$. Since the bystander's best response is weakly decreasing in κ, this implies that $\frac{\partial\Delta}{\partial\kappa} < 0$ at the crossing point $\mu_+ = \mu_+^*$. And $\partial\Delta/\partial\mu_+$ has the same sign as μ_- in general; this can be shown by differentiating Equation (17). It follows that $\frac{\partial\mu_+^*}{\partial\kappa} > 0$ for $\mu_- > 0$ and $\frac{\partial\mu_+^*}{\partial\kappa} < 0$ for $\mu_- < 0$.

[16] Recall that $\partial R/\partial\kappa < 0$, so that the numerator is positive.

Denote a ruler's low and high repression strategies by $\kappa^L = \sqrt{c + (1-\pi)t}$ and $\kappa^H = \sqrt{c}$, respectively. Let $\mu_- > 0$. Then κ^L and κ^H define two curves, $\overline{\mu}_+^L(\mu_-)$ and $\overline{\mu}_+^H(\mu_-)$, which are defined as μ_+^* evaluated at $\kappa = \kappa^L$ and $\kappa = \kappa^H$, respectively. Since μ_+^* is monotonically increasing in κ for $\mu_- \in (0, \infty)$, we have $\overline{\mu}_+^H(\mu_-) < \overline{\mu}_+^L(\mu_-)$ for all $\mu_- > 0$. These two curves define regions of low, intermediate and high repression in the region $\mu_- \in (0, \infty)$.

1. Let $\mu_+ < \overline{\mu}_+^H(\mu_-)$, and let the ruler use cutoff rule $\kappa = \kappa^H$. The bystander's marginal utility of revolt $\Delta(\kappa^H)$ is negative; the bystander's best response is to acquiesce. Since high repression, $\kappa = \kappa^H$, is a best response to the bystander acquiescing, the pair $\kappa^* = \kappa^H$ and $s_b^* = 0$ forms an equilibrium. This is the unique equilibrium in region $\mu_+ < \overline{\mu}_H$.

2. Let $\mu_+ > \overline{\mu}_+^L(\mu_-)$, and let the ruler use cutoff rule $\kappa = \kappa^L$. The bystander's marginal utility of revolt $\Delta(\kappa^L)$ is positive; the bystander's best response is to revolt. Since low repression, $\kappa = \kappa^L$, is a best response to the bystander revolting, the pair $\kappa^* = \kappa^L$ and $s_b^* = 1$ forms an equilibrium. This is the unique equilibrium in region $\mu_+ > \overline{\mu}_L$.

3. For $\overline{\mu}_+^H(\mu_-) < \mu_+ < \overline{\mu}_+^L(\mu_-)$, note that $\Delta(\kappa^H)$ is positive and that $\Delta(\kappa^L)$ is negative. By the intermediate value theorem, there exists $\kappa^\dagger \in (\kappa^H, \kappa^L)$ that makes the bystander indifferent, $\Delta(\kappa^\dagger) = 0$. In turn $\kappa^\dagger$ is a ruler best response to the bystander revolting with probability $\frac{(\kappa^\dagger)^2 - c}{(1-\pi)t}$, such that the pair $\kappa^* = \kappa^\dagger$, $s_b^* = \frac{(\kappa^\dagger)^2 - c}{(1-\pi)t}$ forms an equilibrium. Since the equilibrium is unique, this is the only equilibrium in this region.

The demonstration for the $\mu_- < 0$ region is the same. We define two curves, $\underline{\mu}_+^L(\mu_-)$ and $\underline{\mu}_+^H(\mu_-)$, which are $\mu_+^*(\mu_-)$ evaluated at $\kappa = \kappa^L$ and $\kappa = \kappa^H$ respectively, defined over $\mu_- < 0$. Since μ_+^* is monotonically decreasing in κ, we have $\underline{\mu}_+^L(\mu_-) < \underline{\mu}_+^H(\mu_-)$ for any $\mu_- < 0$. Following the same steps as for $\mu_- > 0$, one can show that the curves $\underline{\mu}_+^L(\mu_-)$ and $\underline{\mu}_+^H(\mu_-)$ define three regions.

1. Let $\mu_+ < \underline{\mu}_+^L(\mu_-)$. The strategies in equilibrium are $\kappa^* = \kappa^L$ and $s_b^* = 1$.

2. Let $\mu_+ > \underline{\mu}_+^H(\mu_-)$. The strategies in equilibrium are $\kappa^* = \kappa^H$ and $s_b^* = 0$.

3. Let $\underline{\mu}_L(\mu_-) < \mu_+ < \underline{\mu}_+^H(\mu_-)$. The strategies in equilibrium are $\kappa^* = \kappa^\dagger$ and $s_b^* = \frac{(\kappa^\dagger)^2 - c}{(1-\pi)t}$.

The last step is to perform the inverse rotation and characterize the five regions in the μ_θ, μ_γ plane.

First, we prove that the function $\mu_+^*(\mu_-)$ is invertible everywhere except at $\mu_- = 0$. We show this by proving that μ_+^* is piecewise monotonic in μ_-. From the implicit function theorem, $\partial\mu_+^*/\partial\mu_- = -\frac{\partial\Delta/\partial\mu_-}{\partial\Delta/\mu_+}\Big|_{\mu_+=\mu_+^*}$. Take the derivative

$$\frac{\partial\Delta}{\partial\mu_-}\Big|_{\mu_+=\mu_+^*} = \left[\mu_+ - \frac{\partial C}{\partial\mu_-}\frac{\partial\ln R}{\partial\kappa} - C\frac{\partial^2\ln R}{\partial\kappa\partial\mu_-}\right]\Bigg|_{\mu_+=\mu_+^*} \tag{19}$$

Note that $\frac{\partial C}{\partial\mu_-} = \mu_+\kappa\left[1 - \tanh^2\left(\frac{\kappa\mu_-}{\sigma^2}\right)\right]$, such that the second term in Equation (19) is positive for $\mu_+ > 0$ and negative for $\mu_+ < 0$. Since $\mu_+^* > 0$ for $\mu_- > 0$ and negative for $\mu_- < 0$, the first two terms are positive when $\mu_- > 0$ and negative when $\mu_- < 0$. We calculate the sign of the third term first by taking the cross-derivative $\frac{\partial^2\ln R}{\partial\kappa\partial\mu_-} = -\frac{\partial}{\partial\kappa}\left(\frac{\partial\ln R}{\partial\kappa}\tanh\left(\kappa\mu_-/\sigma^2\right)\right)$, which gives

$$\frac{\partial^2\ln R}{\partial\kappa\partial\mu_-} = -\underbrace{\frac{\partial^2\ln R}{\partial\kappa^2}}_{<0}\tanh\left(\kappa\mu_-/\sigma^2\right) - \frac{\mu_-}{\sigma^2}\underbrace{\frac{\partial\ln R}{\partial\kappa}}_{<0}\underbrace{\left(1-\tanh^2\left(\kappa\mu_-/\sigma^2\right)\right)}_{>0} \tag{20}$$

Since $\tanh(x)$ is positive for $x > 0$ and negative otherwise, the cross-derivative $\frac{\partial^2\ln R}{\partial\kappa\partial\mu_-}$ is negative for $\mu_- > 0$ and positive for $\mu_- < 0$. Lastly, note that $\Delta = 0 \implies C < 0$, and thus $\mu_+ = \mu_+^* \implies C < 0$. Hence $C\frac{\partial^2\ln R}{\partial\kappa\partial\mu_-}$ is positive whenever $\mu_- > 0$ and negative whenever $\mu_- < 0$. We have shown that $\frac{\partial\Delta}{\partial\mu_-}\Big|_{\mu_+=\mu_+^*}$ is positive whenever $\mu_- > 0$ and negative whenever $\mu_- < 0$.

Take the derivative $\frac{\partial\Delta}{\partial\mu_+} = \mu_- - \frac{\partial\ln R}{\partial\kappa}\sigma^2\tanh(\kappa\mu_-/\sigma^2)$, which is positive whenever $\mu_- > 0$ and negative whenever $\mu_- < 0$. Hence $\frac{\partial\Delta/\partial\mu_-}{\partial\Delta/\partial\mu_+}\Big|_{\mu_+=\mu_+^*} > 0$ and therefore $\partial\mu_+^*/\partial\mu_- < 0$ from the implicit function theorem. Since μ_+^* is monotonic in μ_-, an inverse, $\mu_-^*(\mu_+)$, exists; for any $\mu_+ \neq 0$, we can obtain $\Delta = 0$ by setting $\mu_- = \mu_-^*(\mu_+)$.

Recall that $\mu_\gamma = (\mu_+ - \mu_-)/2$. The function $\mu_+^*(\mu_-)$ defines a mapping $\mu_\gamma^*(\mu_-) = (\mu_+^*(\mu_-) - \mu_-)/2$; for any $\mu_- \neq 0$; we obtain $\Delta = 0$ by setting $\mu_\gamma = \mu_\gamma^*(\mu_-))$. Note that $\mu_\gamma^*(\mu_-)$ is strictly decreasing in μ_- since μ_+^* is strictly decreasing in μ_-; hence $\mu_\gamma^*(\mu_-)$ is invertible. Denote the inverse of μ_γ^* by $g(\cdot)$. This function takes some μ_γ as input, and returns the value of μ_- that makes $\Delta = 0$. Each value of κ defines a function $g(\cdot)$, so we write $g_\kappa(\cdot)$.

Consider the function $\mu_\theta^*(\mu_-) = (\mu_+^*(\mu_-) + \mu_-)/2$. For any $\mu_- \neq 0$, this function returns the value of μ_θ that satisfies $\Delta = 0$ (for any $\mu_- \neq 0$). We can use $g_\kappa(\cdot)$ to express μ_θ^* as a function of μ_γ, i.e., $\mu_\theta^*(\mu_\gamma) = (\mu_+^*(g_\kappa(\mu_\gamma)) + g_\kappa(\mu_\gamma))/2$.

After rotating back to the $\mu_\theta - \mu_\gamma$-plane, the four curves $\underline{\mu}_+^L$, $\underline{\mu}_+^H$, $\overline{\mu}_+^L$ and $\overline{\mu}_+^H$ become

1. $\underline{\mu}_\theta^L(\mu_\gamma) = \frac{1}{2}\underline{\mu}_+^L(g_{\kappa^L}(\mu_\gamma)) + \frac{1}{2}g_{\kappa^L}(\mu_\gamma)$
2. $\underline{\mu}_\theta^H(\mu_\gamma) = \frac{1}{2}\underline{\mu}_+^H(g_{\kappa^H}(\mu_\gamma)) + \frac{1}{2}g_{\kappa^H}(\mu_\gamma)$
3. $\overline{\mu}_\theta^H(\mu_\gamma) = \frac{1}{2}\overline{\mu}_+^H(g_{\kappa^H}(\mu_\gamma)) + \frac{1}{2}g_{\kappa^H}(\mu_\gamma)$
4. $\overline{\mu}_\theta^L(\mu_\gamma) = \frac{1}{2}\overline{\mu}_+^L(g_{\kappa^L}(\mu_\gamma)) + \frac{1}{2}g_{\kappa^L}(\mu_\gamma)$

In other words, we can express the four curves as functions of μ_γ. The rotation preserves the partitioning of the parameter space into regions of low, intermediate and high revolt. High revolt occurs for $\mu_\theta < \underline{\mu}_\theta^H$ and $\mu_\theta > \overline{\mu}_\theta^H$; intermediate revolt occurs for $\underline{\mu}_\theta^H < \mu_\theta < \underline{\mu}_\theta^L$ and $\overline{\mu}_\theta^L < \mu_\theta < \overline{\mu}_\theta^H$; and low revolt occurs for $\underline{\mu}_\theta^L < \mu_\theta < \overline{\mu}_\theta^L$.

Appendix References

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